

On feebly continuous functions and feebly compact space

By : Dr. Saad Naji Al-Azawi⁽¹⁾,
Jamhour Mahmoud Al-Obaidi⁽²⁾
and Aco S. Saied⁽³⁾.

1. University of Baghdad , college of science for women,department of Math.
2. University of Diyala , college of Basic Education , department of Math.
3. Ministry of Higer Education and Scientfic Research studies and planning office

Key words and phrases : feebly - open sets , feebly closed sets ,
feebly–continuous function , f^* feebly – continuous functions ,
 f^{**} feebly–continuous functions and feebly compact spaces
(f-compact space).

Abstract :

The main purpose of this paper is to study a new kinds of continuous functions called feebly continuous function , f^* feebly continuous function and f^{**} feebly continuous functions and study some of their characterizations and proving number of theorems , corollaries , propositions , remarks , examples and their relationships with continuous functions .Also we will introduce a new concept of compact spaces called feebly–compact spaces (f- compact space) and study the relationship of it with usual compact spaces .

1- Introduction :-

The concepts of semi open (*s-open*) and semi closed (*s-closed*) sets are introduced by N. Levine in (1963)[6]. He defined a set A in a topological space X to be *s-open* if for some set G , $G \subseteq A \subseteq \text{cl}(G)$ where $\text{cl}(G)$ denote the closure of G in X . A set is *s-closed* if its complement is *s-open*.

In 1971 S.G. Crossley and S.K. Hildebrand introduced the concept semi closure and they defined it, the semi closure of a set A in a topological space X is the smallest semi closed (*s-closed*) set containing A [3] and denote it by $\text{scl}(A)$. In fact $\text{scl}(A)$ is the intersection of all the semi closed sets containing A , $A \subseteq \text{scl}(A) \subseteq \text{cl}(A)$, $\text{scl}(\text{scl}(A)) = \text{scl}(A)$.

Feebly closed and feebly open set were introduced by Maheswari and Tapi [7] in 1978. A set A in a topological space X is feebly open (*f-open*) set if there exist an open set G such that $G \subseteq A \subseteq \text{scl}(G)$. Every open set is feebly open set but the converse may be false. A set is feebly closed if its complement is feebly open. Every closed set is feebly closed set but the converse may not be true [7]. A function $f : X \rightarrow Y$ is called feebly open (closed) *f-open* (*f-closed*) function if the image of any open (closed) set in X is feebly open (closed) *f-open* (*f-closed*) set in Y , and it is called f^* -open (f^* -closed) function, if the image of each *f-open* (*f-closed*) set in X is open (closed) set in Y , and called f^{**} -open (f^{**} -closed) function, if the image of each *f-open* (*f-closed*) set in X is *f-open* (*f-closed*) set in Y [1].

2- Preliminaries :-

The following are known for any mathematician but they are written because we need them.

Defintion (2-1) [4]

Let X, Y be two topological spaces , A function $f: X \rightarrow Y$ is called continuous at $x_0 \in X$ iff for each open set H in Y containing $f(x_0)$, \exists an open set G in X containing x_0 such that $f(G) \subset H$.

A function is continuous iff it is continuous at each points of X .

Defintion (2-2) [4]

Let X be any set and $T = \{A \in P(X) : A^c \text{ is finite} \}$, then (X, T) is called cofinite topological space ,and denoted by T_c .

Remark (2-3) [5]

A function $f: X \rightarrow Y$ is called *mapping (maps)* if f is continuous.

Theorem (2-4) [5]

Let X, Y be two topological spaces then $f: X \rightarrow Y$ is cotinuous iff the inverse image of every open set in Y is open set in X .

3- Feebly, F^* eebly and F^{**} eebly continuous Functions

In this section we introduce new kinds of continuous functions called feebly continuous function, f^* eebly continuous function, and f^{**} eebly continuous function, and we study their properties and the relationships among them.

(a) Feebly continuous Functions.

Here we give the definition of feebly continous function.

Definition(3-1)

Let X, Y be two topological spaces . A function $f: X \rightarrow Y$ is called feebly continuous (f -continuous) function if the inverse image of every closed(open) set in Y is f -closed (f - open) set in X , denoted by f - continuous mapping .

Example (3-2)

Let X, Y be two topological spaces on X , and (X, T_1) be indiscrete topology on X , (Y, T_2) be any topology on Y , $f: X \rightarrow Y$ be any function, then f is f -continuous function .

proposition : (3-3)

Every continuous function is f -continuous function .

Proof :

Let X, Y be two topological space and $f: X \rightarrow Y$ be continuous function, let A be any open set in Y .
 $\therefore f$ is continuous function, then $f^{-1}(A)$ is open set in X .
 $\therefore f^{-1}(A)$ is feebly open set in X ,
then f is f -continuous function.

□

Remark : (3-4).

The inverse of proposition (3-3) not necessary true as in the following example :

Let $X = \{a, b, c\}$, T_1 is indiscrete topology on X ,
 $T_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ be another topology on X ,
and $g: (X, T_1) \rightarrow (X, T_2)$ defined as : $g(x) = x$.
Then g is f -continuous function, because the inverse image for each open (closed) set in T_2 is feebly open (closed) in T_1 .
but it is not continuous function because the set $\{c\}$ is closed in T_2 , but $g^{-1}\{c\} = \{c\}$ is not closed set in T_1 .

(b) F^* feebly continuous Functions:

Defintion 3-5

Let X, Y be two topological spaces .A function $f: X \rightarrow Y$ is called f^* feebly continuous function (denoted by f^* -continuous) if the inverse image of every feebly open (closed) set in Y is open (closed) set in X .

Example (3-6)

Let X, Y be two topological spaces and T_1 is discrete topology on X .
 T_2 be any topology on Y .
then $f: (X, T_1) \rightarrow (Y, T_2)$ is f^* -continuous function .

Proposition (3-7)

Every f^* -continuous function is continuous .

Proof :-

Let X, Y be two topological spaces and $f: X \rightarrow Y$ be f^* -continuous function .
Let A be any open set in Y , then A is feebly open set in Y ,

$\therefore f$ is f^* -continuous function .
 $\therefore f^{-1}(A)$ is open set in X .
 $\therefore f$ is continuous function.

□

Remark (3-8)

The inverse of proposition (3-7) is not necessary true as in the following example:

Let T_u be usual topology on R , T_i be indiscret topology on R ,
 $f: (R, T_u) \rightarrow (R, T_i)$ be a function defined by :

$$f(x) = 2x \quad \forall x \in R \quad \text{then}$$

- i- f is continuous function because $f^{-1}(R) = R$ and $f^{-1}(\phi) = \phi$ are open (closed) sets in (R, T_u) .
- ii- f is not f -continuous function because if $G = \{2\}$ is feebly open set in T_i but $f^{-1}(G) = \{1\}$ is not open set in (R, T_u) .

(c) F^{} eebly continuous Functions:**

Defintion (3-9)

Let X, Y be two topological spaces , A function $f: X \rightarrow Y$ is called f^{**} eebly continuous function (denoted by f^{**} -continuous) if the inverse image of every feebly open (closed) set in Y is feebly open (feebly closed) set in X .

Example (3-10)

In example (3-6) clearly f is f^{**} -continuous function because the inverse image of any feebly open (closed) set in (Y, T_2) is feebly open (closed) set in (X, T_1) .

proposition(3-11)

Every f^* -continuous function is f -continuous .

Proof :-

By the same method in proving propositions (3-3) and (3-7).

□

Remark (3-12) :

The inverse of proposition (3-11) is not necessary true as in the following example :

in remark(3-4) , hence g is f -continuous function because g^{-1} for each closed set in T_2 is feebly closed set in T_1 , and it is not f^* -continuous function because $\{c\}$ is feebly closed set in T_2 but $g^{-1} \{c\} = \{c\}$ is not closed set in T_1 .

proposition (3-13)

Every f^* -continuous function(mapping) is f^{**} -continuous function .

Proof :-

By the same method in proveing propositions (3-3) and(3-7).

□

Remark (3-14)

The inverse of proposition (3-13) is not necessary true as in the following example .

Let T_u be usual topology on R , T_i be indiscret topology on R ,
 $f : (R ,T_i) \rightarrow (R ,T_u)$ be a function defind by :

$$f(x) = x \quad \forall x \in R \quad \text{then}$$

- i- f is f^{**} -continuous function because the inverse image of any feebly closed set in (R,T_u) is feebly closed set in (R,T_i) .
- ii- f is not f^* -continuous function because if $G = (a,b)$ is feebly open set in (R,T_u) but $f^{-1}(G) = (a,b)$ is not open set in (R ,T_i) .

proposition (3-15)

Every f^{**} -continuous function is f -continuous function.

Remark(3-16)

The inverse of proposition (3-15) is not necessary true as in the following example:

Let $X = \{a,b,c\}$, $T_1 = \{\phi , X, \{a\} , \{a,b\}\}$ be topology on X .

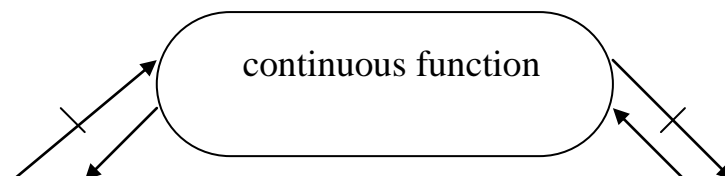
and $T_2 = \{ \phi , X \}$ is the indiscrete topology on X .

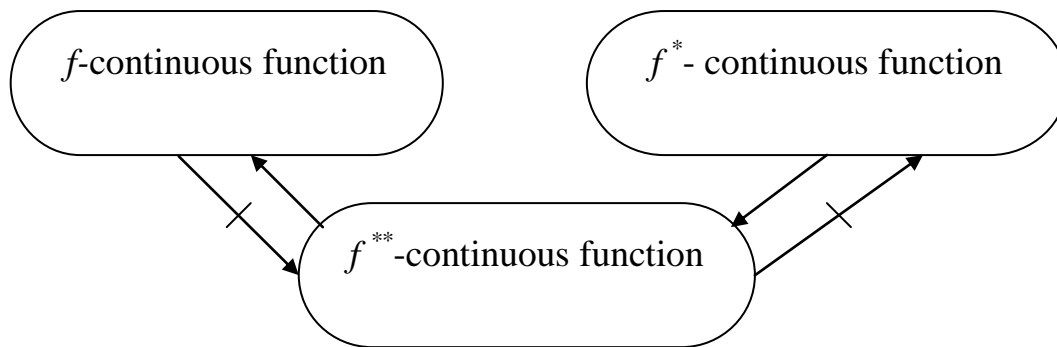
$f : (X,T_1) \rightarrow (X,T_2)$ defined by $f(x) = x$.

then :

- i- f is f -continuous function because the inverse image of any closed set in T_2 is feebly closed set in T_1
 i.e $f^{-1}(\phi) = \phi$, $f^{-1}(X) = X$ are feebly closed set in T_1 .
- ii- f is not f^{**} -continuous function because $\{b\}$ is feebly closed set in T_2 but $f^{-1}\{b\} = \{b\}$ is not feebly closed in T_1 .

The following diagram shows the relation among continuous function, f -continuous function, f^* -continuous and f^{**} - continuous function.





4- The main Results.

feebly compact space (f-compact space)

In this section we define a new concept of compactness called feebly compact space (f-compact space) and we will study it's relationship with usual compact space .

Definition (4-1):

Let (X,T) be a topological space , $A \subseteq X$.

A family $\{G_i : i \in \Delta\}$ of subsets of X is cover for A iff $A \subseteq \bigcup_{i \in \Delta} G_i$,

and it's called open cover if the elements of the cover are open.

Definition (4-2):

Let a family $\{G_i : i \in \Delta\}$ be an open cover of a set A , then if their exist finite subset Δ_o of Δ such that

$$A \subseteq G = \bigcup_{i \in \Delta_o} \{G_i : i \in \Delta_o\} \text{ then } G \text{ is called finite subcover of } A.$$

Definition (4-3):

Let (X,T) be a topological space , X is called compact space iff every open cover of X has finite subcover.

Remark(4-4):

Let (X,T) be a topological space , $A \subseteq X$ then A is called compact subset on X , if every open cover of A has finite subcover.

Example (4-5):

Let (N,T_c) be cofinite topology on N , then (N,T_c) is compact space because :

If $U = \{U_i : i \in \Delta\}$ any open cover of N , let U_{i_0} any set in U then $N-U_{i_0}$ is finite set , let $N-U_{i_0} = \{n_1, n_2, \dots, n_p\}$ then their exist finite sets $U_{i_1}, U_{i_2}, \dots, U_{i_p}$ hence $n_k \in U_{i_k}$, $k = 1, 2, \dots, p$ then the family $\{ U_{i_1}, U_{i_2}, \dots, U_{i_p} \}$ is finite subcover of N ,

Hence (N, T_c) is compact space.

Now we present the definition of feebly compact (f-compact) space after introducing the concept of feebly open cover (f – open cover).

Definition (4-6):

A family $\{V_i : i \in \Delta\}$ of feebly open sets (f-open sets) in (X, T) is called feebly open cover (f- open cover) of A iff $A \subseteq \bigcup_{i \in \Delta} V_i$.

Definition (4-7):

A topological space (X, T) is called feebly compact (f-compact) space iff every f-open cover of X has finite subcover .

Remark (4-8):

In a topological space (X, T) , $A \subseteq X$ then A is called feebly compact subset (f-compact subset) on X , if every f-open cover of a set A in X has finite subcover.

Example (4-9):

The topological space in example (4-5) is f-compact space because the f-open set in N are ϕ, N only.

Now we will study the relationship between compact and f-compact spaces and we will find that they are independent concepts as in the following examples:

Example (4-10):

Let (R, T_i) be the indiscrete topology on R , (R, T_i) is compact space but it is not f-compact space , because every subset of R is f-open set then the family of singleton set in R $W = \{x\} \forall x \in R$ is f-open cover of R but W hasn't finite subcover.

Example (4-11):

Let (R, P) be a topological space defined on R as :
 $P = \{\phi\} \cup \{A \subseteq R : 0 \in A\}$. Then (R, P) is not compact space because the open cover $G = \{(-n, n) : n = 1, 2, 3, \dots\}$ haven't finite subcover of R , but (R, P) is f-compact space because the f-open set in R are ϕ, R only.

Theorem (4-12): [2]

Let $f : X \rightarrow Y$ be continuous and onto function, if X is compact space then Y is compact space .

Now we will study the f-compact properties

Theorem (4-13):

Let $f : X \rightarrow Y$ be f -continuous and onto function if X is f-compact space then Y is compact space .

Proof: Let $\{K_i : i \in \Delta\}$ is open cover of Y .

$\therefore f$ is f - continuous function (mapping).

$\therefore \{f^{-1}(K_i) : i \in \Delta\}$ is f-open cover of X .

$\therefore X$ is f-compact space.

$\therefore X$ has finite subcover $\{f^{-1}(K_1), f^{-1}(K_2), \dots, f^{-1}(K_n)\}$

$\therefore f$ is onto function (mapping) .

$\therefore f(f^{-1}(K_i)) = K_i, \forall i \in \Delta$

$\therefore \{K_1, K_2, \dots, K_n\}$ is finite subcover of Y .

$\therefore Y$ is compact space.

□

Theorem (4-14):

Let $f : X \rightarrow Y$ be f^* -continuous and onto function , if X is f-compact space then Y is f-compact space.

Proof:

Let $\{G_i : i \in \Delta\}$ is f-open cover of Y ,

$\therefore f$ is f^* -continuous function,

then $\{f^{-1}(G_i) : i \in \Delta\}$ is open cover of X ,

then $\{f^{-1}(G_i) : i \in \Delta\}$ is f-open cover of X ,

(every open set is f-open set)[def 2-3]

$\therefore X$ is f-compact space, $\therefore X$ has finite subcover

$\{f^{-1}(G_1), f^{-1}(G_2), \dots, f^{-1}(G_n)\}$

$\therefore f$ is onto function then $f(f^{-1}(G_i)) = G_i : \forall i \in \Delta$

$\therefore \{G_1, G_2, \dots, G_n\}$ is f-open cover of Y .

$\therefore Y$ is f- compact space.

□

Theorem (4-15):

Let $f : X \rightarrow Y$ be f^{**} -continuous and onto function , if X is f -compact space then Y is f -compact space.

Proof :

Let $\{G_i : i \in \Delta\}$ is f -open cover of Y ,
 $\because f$ is f^{**} -continuous function,
then $\{f^{-1}(G_i) : i \in \Delta\}$ is f -open cover of X ,
 $\because X$ is f -compact space, $\therefore X$ has finite subcover
 $\{f^{-1}(G_1), f^{-1}(G_2), \dots, f^{-1}(G_n)\}$
 $\because f$ is onto function then $f(f^{-1}(G_i)) = G_i : \forall i \in \Delta$
 $\therefore \{G_1, G_2, \dots, G_n\}$ is f -open cover of Y .
 $\therefore Y$ is f - compact space.

□

Remarks (4-16):

1- Let X, Y be two topological spaces and let $f : X \rightarrow Y$ be function , if A is compact subset of X then:

- i- $f(A)$ is compact subset of Y , if f is open function and bijective function .
- ii- $f(A)$ is f - compact subset of Y , if f is f -open function and bijective function .

2- Let X, Y be two topological spaces and let $f : X \rightarrow Y$ be function , if A is f - compact subset of X then:

- i- $f(A)$ is compact subset of Y , if f is f^* -open and bijective function .
- ii- $f(A)$ is f - compact subset of Y , if f is f^{**} - open and bijective function .

References :-

- 1- AL.Obaidi J.Mahmoud ***On feebly closed functions***
To appear in Al-fath , Journal Diyala University .
- 2- Al. Mayahy F.Jameel ***On θ - G - closed Mappings***
M.sc. thesis Al - Mustansiriya University 2002.
- 3-Das.P, ***Note on some applications on semi open sets***
progress of Math, Allahabad University, India.
- 4- Hu. S.T. ; ***Element of General Topology,***

- Hadden .Day Inc. San Francisco,(1965).
- 5- Husain. T . **Topology and Maps** ,Mathematical concepts and Methods in Science and Engineering,(1977).
- 6- Levine.N **semi open set and semi continuity in topological spaces** . Amer. Math.Monthly,(1963).
- 7- Maheshwari S.N & Tapi U. **Feebly open sets** . Ann , Univ . Timisoaras. (1978).
- 8- Navalagi G.B **(Definition bank) in General topology**.Internet (2000).
- 9- Steen .L.A. ; **Counter examples in Topology**,(1970).

حول الدوال المستمرة الواهنة والفضاءات المرصوفة الواهنة

أ.م.د. سعد ناجي العزاوي (مؤلف) ، م.م. جمهور محمود العبيدي (مترجم) ، أكو سعيد (مترجم)

- 1 - جامعة بغداد - كلية العلوم للبنات - قسم الرياضيات
- 2 - جامعة ديالى - كلية التربية الأساسية - قسم الرياضيات
- 3 - وزارة التعليم العالي والبحث العلمي - دائرة التخطيط والدراسات

المستخلص:

سيتم في هذا العمل دراسة أنماط جديدة من الدوال المستمرة تسمى الدوال المستمرة الواهنة (*feebly continuous*) ، الدوال المستمرة

الواهنة * (f^* eebly continuous) والدوال المستمرة
الواهنة ** (f^{**} eebly continuous) ودراسة بعض خواص هذه
الأنماط وبرهنة عدد من المبرهنات والنتائج والقضايا الخاصة بذلك
واعطاء عدد من الملاحظات والأمثلة ودراسة علاقتها مع الدوال
المستمرة . وكذلك تقديم صيغة جديدة من صيغ الفضاءات المرصوصة
تسمى الفضاءات المرصوصة الواهنة (f -compact space)
ودراسة علاقتها مع الفضاءات المرصوصة الاعتيادية وبرهنة بعض
المبرهنات والملاحظات والأمثلة التي تبين تلك العلاقة.