Solve Phase Queuing Models by Using Markov Transition Rate Matrix

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Abstract:

Markov processes are considered an appropriate for queuing system. As they suppose that unites arrival process to the system followed a certain distribution and the service times should be separate form each others. This means that each time is considered to be a random variable having a certain probable distribute. Many researchers concern for developing and founding new methods to solve complex queuing models. This research will take many articles to solve phase distributed queuing models the first article is Birth-Death process. The second is Quasi-Birth-Death process. The third are illustrates the Level Dependent Quasi Birth –Death Process (LDQBD), while the forth concern on the numerical solve and results testing. Last and fifth article discuses the main conclusions reached by depending on Transition Rate Matrix.

Aim from the Research:

To found numerical and analytic phase distributions models using transition rate matrix, then solve deferential equations with numerical or analytic style.

1- INTRODUCTION

This article presents the necessary theory for **LDQBD's** and briefly comments upon the new algorithms for calculating $\mathbf{R_k}$ and $\mathbf{G_k}$. First we successively give examples to be modeled by a brief sophisticated Markov Process and reasons why the extra sophistication is necessary. The analytical solutions to these models and what these solutions mean is then presented which is followed by a brief comment on some new techniques for their numerical solution. The paper concludes with results obtained by applying the numerical techniques to examples. These results test the accuracy of the new solutions.

2- A BIRTH- DEATH PROCESS:

The birth-death process is a special case of a Markov process and encompasses a wide range of models such as the Poisson process and the **M/M/I** queue. In t his section we will define a birth-death process and give an example of its application.

Definition 1 The continuous-time Markov $\mathbf{\{X\{t) : \mathit{t} > 0\}}$ is a **birth—**

death process if the only two possible transition are **n→ n+1** with birth rates **q(n,n+1), n ≥ 0,** and **n→ n-1** with death rates **q(n,n-1), n ≥ 1**. The standard infinitesimal generator matrix for a birth-death process is given by:

$$Q = \begin{bmatrix} -q(0) & q(0,1) & 0 & 0 & . & . & . \\ q(1,0) & -q(1) & q(1,2) & 0 & . & . & . \\ 0 & q(2,1) & -q(2) & q(2,3) & . & . & . \\ 0 & 0 & q(3,2) & -q(3) & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \qquad \ldots\ldots.(1)$$

Where

q(j) = q(j,j-1) +q(j,j+1), $\forall$ ***j*** $\in$ ***s***. as the process can not skip adjacent states , we say the process is "skip free" in the states. The equilibrium equations of a birth -death process **are** ***π(j) q(j,j -1) = π (j-1) q(j-1 ,1)*** and are satisfied by The equilibrium distribution given by

$$\Pi(j) = \Pi(0)\prod_{r=1}^{j}\frac{q(r-1,r)}{q(r,r-1)} \qquad \ldots\ldots(2)$$

the following example illustrates a special case of the birth- death process, namely , the **M/M/1** queue .

Example: 1 Suppose that the arrival process, that is the stream of customers arriving at a queue, forms a Poisson process of rate **λ**. suppose further that there is a single server and that customers' service times are independent of each other and of the arrival process and are exponentially distributed with mean μ^{-1}**.** Such a queue is called the simple or **M/M/l** queue, the **M's** indicating the memory less (exponential) character of the inter-arrival and service times and the final digit indicating the number of servers. Let **X(t)** be the number of customers in the system at time ***t***. Then it follows from Definition 1 that, ***X(t)*** is a birth-death process with transition rates

$$q(j,j+1)=\mu \, , \, j \in s \qquad \ldots(3)$$

$$q(j,j-1)=\lambda \, , \, j \in s \qquad \ldots(4)$$

The process can be represented by the state transition diagram in **fig.1** . Where the number in the boxes

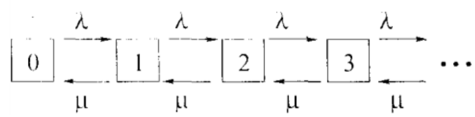

Fig.1 the state transition diagram

Represents the possible states, and the values by the arrows represents the rate of transition between the stares in the direction indicated.

This can be modeled by a birth-death process with the following infinitesimal generator matrix derived from equations **(1),(3)** and **(4)** .

$$Q=\begin{bmatrix} -\lambda & \lambda & 0 & 0 & . & . & . \\ \mu & -(\lambda+\mu) & \lambda & 0 & . & . & . \\ 0 & \mu & -(\lambda+\mu) & \lambda & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

2.1. Limitation of the Birth –Death model

Birth- death processes work well in modeling situations that are skip free in the states, such as customers in a line or a fish population in a lake. There are, however, many situations with the Markov property where the birth death process is inadequate.

Example: 2 Customers arrive at a service unit, according to a Poisson process of rate **λ.** Services occur in groups, with the group size dependent on the queue length according to the following rule. Let there be *i* customers waiting at the completion of a service.

If **0 < i <** ***L***. the server remains idle until the queue length reaches ***L,*** and then starts serving all ***L*** customers. If ***L*** **< i < m.** a group of size **i** enters service, and it ***i***>***m***, a group of size ***m*** is served. It is assumed that the lengths of service of successive groups are conditionally independent.

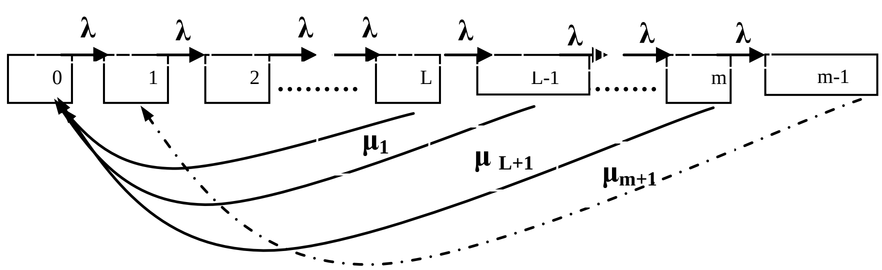

Fig.2: State Transition Diagram for Example 2.

The bivariate sequence of the queue lengths following departures and the times between departures defines a Markov process. However, by examining the state transition diagram ill **Fig. 2** it can be seen that the transitions are not skip tree in the states, so tins situation can not be modeled by a birth- death process (see **Definition 1**). In the next section we will see how the problem of modeling this situation and many others- was overcome.

3-QUASI-BIRTH-DEATH-PROCESSES

The following definition was formed form the explanation given as:

Definition 2 -4 continuous time **Quasi—Birth-Death (QBD)** process is a continuous time Markov process whose infinitesimal generator matrix is of the block partitioned form

$$Q = \begin{bmatrix} Q_1^{(-1)} & Q_0 & 0 & 0 & . & . & . \\ Q_2 & Q_1 & Q_0 & 0 & . & . & . \\ 0 & Q_2 & Q_1 & Q_0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \quad \text{.... (5)}$$

After partitioning the states into subsets $l(i) = \{(i,j) : i \geq 0, 1 \leq j \leq m\}$ called levels, position **j** within the level is termed the **phase.** The process can jump down one level, stay in the same level or jump up one level and the rate that these transitions occur are given by the **m*m** matrices $\mathbf{Q_1, Q_2}$, and $\mathbf{Q_0}$ respectively. The process is said to be skip free between levels.

The trick with modeling **Example.2** is to redefine the state space so that we have levels that are skip free. This can be done by making the levels successive sets of ***m*** customers, then the possible state transitions have been transitions to that represented by **Figure .1** But describing the number of customers in the system has now become two dimensional, because, Although ***in*** customers are served at a time, they arrive one at a time. This means that when there are ***i***

.m + j customers in the system, it is represented by the state **(i,j),** where $0 \le j \le m-1$. Incorporating this state space with the elements of **Q** in equation **(5)** gives the following sub-matrices.

$$Q_1^{(-1)} = \begin{bmatrix} S_1 & 0 \\ S_2 & S_3 \end{bmatrix} \qquad \text{……..(6)}$$

Whose components are defined by :

$$S_1 = \begin{bmatrix} -\lambda & \lambda & 0 & . & . & . & 0 & 0 \\ 0 & -\lambda & \lambda & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -\lambda & \lambda \\ 0 & 0 & 0 & . & . & . & 0 & -\lambda \end{bmatrix} \qquad \text{…… (7)}$$

An+ L*L matrix.

$$S_1 = \begin{bmatrix} \mu_L & 0 & 0 & . & . & . & 0 & 0 \\ \mu_{L+1} & 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \mu_m & 0 & 0 & . & . & . & 0 & 0 \end{bmatrix}, \qquad \text{….. (8)}$$

an **(m-L+1)*L** matrix with the only non zero column being the first, and

$$S_3 = \begin{bmatrix} -\lambda & \lambda & 0 & . & . & . & 0 & 0 \\ 0 & -\lambda & \lambda & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -\lambda & \lambda \\ 0 & 0 & 0 & . & . & . & 0 & -\lambda \end{bmatrix}, \qquad \text{…….. (9)}$$

an ***(m - L + 1)* x (m - *L* + 1)** matrix. The three matrices $\mathbf{Q_2}$**.** $\mathbf{Q_1}$ and **Qo** that repeat throughout, **Q** in equation **(5)** for this example are all ***m* x *m*** and are defined as follows.

$$Q_2 = \begin{bmatrix} \mu_m & 0 & 0 & . & . & . & 0 & 0 \\ 0 & \mu_m & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & \mu_m \end{bmatrix} \quad \ldots\ldots\ldots (10)$$

$$Q_1 = \begin{bmatrix} -\lambda & \lambda & 0 & . & . & . & 0 & 0 \\ 0 & -\lambda & \lambda & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -\lambda & \lambda \\ 0 & 0 & 0 & . & . & . & 0 & -\lambda \end{bmatrix} \quad \ldots\ldots\ldots (11)$$

And Q_0 is m*m with the only non-zero entry being λ in the bottom left hand entry.

3.1. *Limitations of the Quasi- Birth- Death model.*

The Quasi-Birth Death model effectively models the behavior of processes that are skip free in the levels if the transition rates are independent of the level the process is in (after the first level). Such models are encountered in areas such as telecommunications and manufacturing processes. However, there are many situations, (in biological systems for instance) where the transition rates are most definitely dependent upon the level the system is in, and Quasi-Birth-

Death processes become inappropriate to model the behavior. The following simple example illustrates this point.

Example: 3 consider a colony of algae (referred to as a population). At any given time, each member of the population can either give birth to one new member, or it can die. The temperature affects rates of birth and death within the population, and to model the affect of this factor, it has been quantified to 2 different phases, where 1 is cold and 2 is hot. Assuming the temperature is uniformly distributed throughout the population, the new birth and death rates for an individual are

$$\lambda_j=\lambda(j) \qquad ....(12)$$

$$\mu_j=\mu(j)$$

Where $\lambda(j)$ and $\mu(j)$ are functions that give the birth and death rates at temperature j respectively. The other possible events that can occur arc a change in temperature. We will assume that these changes occur instantaneously at rate γ_1 from cold to hot and rate γ_2 from hot to cold. All events are considered to occur separately. Two things need to be represented in the state description, namely, the number of algae present n, and the temperature factor of the environment j, so a point (n,j) in the state space $S = \{\{n, j). n \in Z^+, j \in \{1.2\}\}$ represents a population size of n algae living at a temperature of j. Because each individual in the population can give birth or die, as the population size increases, the birth and death rates increase. Thus the rates for the system with population n and temperature $\boldsymbol{j}$ are given by

$$\lambda_{nj} = n\lambda(j) \qquad \text{......(13)}$$

$$\mu_{nj} = n\mu(j)$$

The **QBD** process can cope with modeling the behavior of a population given that the transition rates change with a factor (such as temperature), as it is defined with a two dimensional state space, However, the model becomes inappropriate when the transition rate changes due to the level (in this case the number of algae present). Overcoming this problem naturally leads to the next section.

4-LEVEL DEPENDENT QUASI BIRTH-DEATH PROCESSES (LDQBD)

A level dependent ***QBD*** differs from a ***QBD*** in that the transition rates and the number of phases at each level can be dependent upon the level the process is in. The following definition is adapted from Bright..

Definition 3 .A continuous times Level Dependent Quasi Birth Death **(LDQBD)** is a, continuous time-two dimensional Markov process $X(t)$ on the state space $S = \{(k,j);\ k>0,\ \ 1< j < M(k)\}$ with infinitesimal generator of the blocked partitioned form

$$Q = \begin{bmatrix} Q_1^{(0)} & Q_0^{(0)} & 0 & 0 & . & . & . \\ Q_2^{(1)} & Q_1^{(1)} & Q_0^{(1)} & 0 & . & . & . \\ 0 & Q_2^{(2)} & Q_1^{(2)} & Q_0^{(2)} & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \qquad \text{……….(14)}$$

where $Q_0^{(k)}, k \geq 0$ $Q_1^{(k)}, k \geq 0$, $Q_2^{(k)}, k \geq 1$ are matrices of order ***M(k)xM(k+1), M(k)xM(k)*** *and* ***M(k)xM(k-1)*** and given the rates of going up one level , staying in the same level or going down one level respectively. We say the process is skip free in the levels.

A **LDQBD** can model the situation in **Example 3** using the infinitesimal generator matrix given in equation **(14)** with the population size defined as levels. and the temperature as phases within each level. It should be noted here that it is assumed that the process starts with at least one member in the population (ie; in level **1**), as this is necessary for the population to propagate. In the following equations the subscript indicates the temperature phase. Thus for

$n > 1$

$$Q_2^{(n)} = \begin{bmatrix} n\mu(1) & 0 \\ 0 & n\mu(2) \end{bmatrix},$$

$$Q_1^{(n)} = \begin{bmatrix} -(n(\mu(1)+\lambda(1))+\gamma_1 & \gamma_1 \\ \gamma_2 & -(n(\mu(2)+\lambda(2))+\gamma_2 \end{bmatrix},$$

And

$$Q_0^{(n)} = \begin{bmatrix} n\lambda(1) & 0 \\ 0 & n\lambda(2) \end{bmatrix}.$$

Also $\mathbf{Q_i^{(0)}} = \boldsymbol{0}$ for **i = 0,1,2**, since, if the population dies out it will not recover. Thus level zero is called ***absorbing*** as the process will remain in this level if it is entered.

5- DETERMINING RESULTS FOR A LDQBD

Meaningful results can be obtained from this mathematical model in terms of two matrices called $\boldsymbol{G_k}$, and $\boldsymbol{R_k}$. An interpretation of these matrices and how they are used is outlined in the following.

A. The G_k, Matrix

The $\boldsymbol{G_k}$. matrix for a **LDQBD** gives the first passage probabilities downward to each level from the immediately higher level. The **(i,j)-th** element of the matrix $\boldsymbol{G_k}$ **,** is the probability that. Starting in the state **(k,i)**, the process reaches level ***k-1*** in finite time, and does so first through the state ***(k-1,j)***. An example of a sample path that the probability $|\boldsymbol{G_k}|_{ij}$ includes is given in **Figure .3**

B. An. Expression, for G_k,

The matrix $\boldsymbol{G_k}$ can be expressed as the minimal nonnegative solutions to a family to non-linear matrix equations. Prove that for a continuous time **LDQBD** the family of matrices **{** $\boldsymbol{G_k}$ **,k>1 }** are the minimal nonnegative solution to the matrix equation

$$\boldsymbol{Q_2^{(k)} + Q_1^{(k)} G_k + Q_0^{(k)} G_{k+1} = 0, k \geq 1} \qquad \text{...... (15)}$$

Where $Q_i^{(k)}$, . $i.$ = 0, 1, 2 are the sub-matrices of the transition matrix as given in Definition 3.

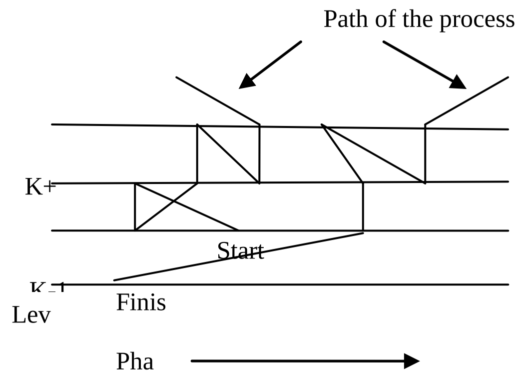

Fig. 3: The Process Path Attributed to $|G_k|_{i,j}$

C.The R_k Matrix

The family of matrices R_k is used to find the stationary distribution of a **LDQBD**. The physical interpretation distribution of this matrix is briefly outlined in the following. For a more detailed description of R_k.

Theorem 1: For appositive recurrent continuous time **LDQBD**

$$\pi_k = \pi_{k-1} R_{k-1}, \quad k \geq 1 \qquad \text{....... (16)}$$

Where π_k is the vector of stationary probabilities of being in one of the phases in state k , and $[R_{k-1}]_{j,j}$ is the expected ***sojourn time*** in the state **(k,j),** in units of the mean ***sojourn time*** in the state **(k-1,i)** , before returning to level **k-1** , given the process starts in the **(k-1,i)** .

D. An Expression for R_k

The family of $\mathbf{R_k}$ matrices satisfy a non-linear matrix equation. Ramaswami discusses $\mathbf{R_k(s)}$ for a continuous time Markov process with a **2** dimensional state space that is skip free upwards in the levels, where $\mathbf{R_k(s)}$ is the generating function form of $\mathbf{R_k}$. Generalizing Ramaswami's results for a **LDQBD** process, $\mathbf{R_k(s)}$ is given by

$$R_k(s) = Q_0^{(k-1)}\int_0^{\infty} e^{-st}k-1P(k,k;t)dt, \quad k \geq 1 \qquad \text{........ (17)}$$

And satisfy the equations

$$sR_k(s) = Q_0^{(k-1)} + R_k(s)Q_1^{(k+1)} + R_k(s)R_{k+1}(s)Q_2^{(k+1)} \qquad \text{......... (18)}$$

Now taking ***s*** to bo zero we get that $\mathbf{R_k}$ satisfies

$$Q_0^{(k)} + R_kQ_1^{(k+1)} + R_k[R_{(k+1)}Q_2^{(k+2)}] = 0, \quad k \geq 0 \qquad \text{......... (19)}$$

$\mathbf{R_k}$ is the minimal non-nagative solution to the above equation, Whose solutions are not necessarily unique.

E. The Relationship between R_k and G_k

In a continuous time **LDQBD**, the elements of and *Gi*, both give a measure referring to a return level while only visiting higher levels in between important difference to note between the two matrices is that $[Ri.]_{iJ}$ is a relative time and $[Gi]_{i,j}$ is a probability. There is a relationship between these matrices however which is proved in Bright .

Theorem 2 The family of matrices $\{G_k$, k >=1} and $\{R_K$,. k >= 0} satisfy the following equations:

$$G_k = (-Q_1^{(k)} - R_kQ_2^{(k+1)})^{-1}Q_2^{(k)}, \qquad \text{..........(20)}$$

$$R_k = Q_0^{(k)}(-Q_1^{(k+1)} - Q_0^{(k+1)}G_{k+2})^{-1}, \qquad ....(21)$$

6- THE NUMERICAL SOU'TION

Latouche and Ramaswami presented a major break-through in matrix-analytic methods when they devised the Logarithmic Reduction algorithm to calculate the ***R*** and ***G*** matrices for a ***QBD*** process. This algorithm converged faster than previous algorithms and had good numerical stability characteristics, and recently Bright and "Taylor adapted this Logarithmic reduction algorithm to calculate R_k and G_k, for a ***LDQBD***. Thorne discovered that these algorithms used memory exponentially with the complexity of the problem, and constructed a new recursive algorithm to improve the memory usage of this algorithm. The following shows the results of applying these new algorithms to examples and tests to show their accuracy.

1- *Testing the Results*

Once tlie matrices G_k and R_k, were determined numerically, a way of validating them was needed. R_k is the minimal non-negative solution to the system of equations (19). Rearranging equation (19) gives

$$R_k = (-Q_0^{(k)})[Q_1^{(k+1)} + R_{k+1}Q_2^{(k+2)}]^{-1}, \qquad .... (22)$$

Thus we can find the value of R_k from R_{k+1} , Similarly, by rearranging equation (15) we can get the following expression for G_k in terms of G_{k+1} .

$$G_k = [Q_1^{(k)} + Q_1^{(k)}G_{k+1}]^{-1}(-Q_2^{(K)}), \qquad .... (23)$$

Because the solutions to these equations are not unique, the relation between G_k and R_k given by equation (20) was used to further results.

2- *Results*

The code for the new algorithm was tested with the ***LDQBD*** defined for the algae population in Example 3 . If the population of algae is 100, the probability that the population of 99 algae will ever recur needs to be known. The matrix $[G_{100}]$ can answer this question as $[G_{100}]_{I,J}$ gives the probability that starting with a population of 100 at temperature j the population eventually drops down to 99 algae at a temperature j . The matrix R_{100} is used to validate G_{100} . For this model, temperature is factored into 7 different phases and the necessary parameters were defined *as* follows.

- λ =1 for the individual birth rate.
- μ = 2 for the individual death rate.
- γ = 0.5 for all changes in temperature.

The following output was obtained.

The matrix R k = 100

$$\begin{bmatrix} 49*10^{-2} & 45*10^{-4} & 11*10^{-5} & 57*10^{-7} & 41*10^{-8} & 4*10^{-8} & 0 \\ 22*10^{-4} & 48*10^{-2} & 66*10^{-4} & 21*10^{-5} & 12*10^{-6} & 10*10^{-7} & 11*10^{-8} \\ 39*10^{-7} & 44*10^{-4} & 48*10^{-2} & 85*10^{-4} & 34*10^{-5} & 22*10^{-6} & 22*10^{-7} \\ 14*10^{-7} & 11*10^{-5} & 64*10^{-4} & 47*10^{-2} & 10*10^{-3} & 47*10^{-5} & 39*10^{-6} \\ 8*10^{-8} & 51*10^{-7} & 21*10^{-5} & 8*10^{-3} & 47*10^{-2} & 12*10^{-3} & 68*10^{-5} \\ 1*10^{-8} & 34*10^{-8} & 12*10^{-6} & 32*10^{-5} & 10*10^{-2} & 47*10^{-2} & 14*10^{-3} \\ 0 & 3*10^{-8} & 98*10^{-8} & 22*10^{-6} & 48*10^{-5} & 12*10^{-3} & 48*10^{-2} \end{bmatrix}$$

As can be seen from the output, the matrices generated by the improved memory efficient algorithm are consistent with those calculated by equations (22) and (20), The G_{100} matrix implies that, the probability of a return to a population of 99 is very high concentrated around the concurrent event that the temperature is the same as the initial temperature.

7- CONCLUSION

This project has been concerned with the presentation of R_K, and G_k and implementation of an adaptation of the Logarithmic Reduction algorithm Bright to evaluate them in a LDQBD process. The results suggest that the more memory efficient version of these algorithms implemented by Thorne are accurate, though further analysis needs to be done to show how much memory they save.

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الخلاصة :

تعد عمليات ماركوف ملائمة لنظام صفوف الأنتظار لأنها تفترض أن تكون عملية وصول الوحدات إلى النظام تتبع توزيع معين، وأن يكون زمن الخدمة مستقل الواحد عن الأخر ، أي هو عبارة عن متغير عشوائي له توزيع أحتمالي معين . فقد أهتم الكثير من الباحثون في تطوير وإيجاد أساليب جديدة لحل نماذج صفوف الأنتظار المعقدة ، لذلك سيناول بحثنا هذا عدة فقرات لإيجاد حل لنماذج صفوف الأنتظار ذات التوزيعات الطورية ، الأولى منها عملية الولادة –وفاة Birth-Death process ، والفقرة الثانية شبه عمليات ولادة–وفاة Quasi- Birth-Death process ، والفقرة الثالثة تم بيان فيها مستوى الأعتماد على شبه عمليات الولادة –وفاة Level Dependent Quasi- Birth-Death process وبأختصار (LDQBD) ، والفقرة الرابعة تناولت الحل العددي وأختبار النتائج، أما الفقرة الخامسة والأخيرة تناولت أهم الأستنتاجات التي تم التوصل أليها من خلال أعتماد مصفوفة معدل الأنتقال Transition rate matrix .