Stability of the Finite Difference Methods of Fractional Partial Differential Equations Using Fourier Series Approach

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Abstract:

The fractional order partial differential equations (FPDEs) are generalizations of classical partial differential equations (PDEs).

In this paper we examine the stability of the explicit and implicit finite difference methods to solve the initial-boundary value problem of the hyperbolic for one-sided and two sided fractional order partial differential equations (FPDEs). The stability (and convergence) result of this problem is discussed by using the Fourier series method (Von Neumann′s Method).

استقرارية معادلات الفروق المنتهية للمعادلات التفاضلية الكسرية الجزئية باستخدام اسلوب متسلسلات فوريه

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المستخلص:

ان المعادلات التفاضلية الكسرية الجزئية هي تعميم للمعادلات التفاضلية الجزئية.

وفي هذا البحث استخدمت طريقة الفروقات المنتهية الصريحة والضمنية لحل مسألة القيم الابتدائية والحدودية للمعادلات التفاضلية الكسرية الجزئية ذات الجهة الواحدة وذات الجهتين.

وقد نوقشت نتائج الاستقرارية والتقارب لهذه المسألة باستخدام متسلسلات فوريه.

1.Introduction

In this paper, we are going to modify a new approach for investigating the stability of the FPDEs by the Fourier series method (Von Neumann′s method).

We will study the simplest form of hyperbolic PDE of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial^q u(x,t)}{\partial x^q},\ L \le x \le R,\ 0 \le t \le T. \qquad \ldots[1]$$

where q is the fractional numerical and $1 \le q \le 2$.

Together with the initial and zero Dirichlet boundary conditions:

$$\left.\begin{aligned} &u(x,0)=f(x) && \text{for} \quad L \le x \le R \\ &\frac{\partial u(x,0)}{\partial t}=g(x) && \text{for} \quad L \le x \le R \\ &u(L,t)=0 && \text{for} \quad 0 \le t \le T \\ &u(R,t)=0 && \text{for} \quad 0 \le t \le T \end{aligned}\right\} \qquad \ldots[2]$$

We use the explicit and implicit finite difference methods to solve eq.[1] for one-sided FPDEs. Also, to solve two-sided FPDEs that the following form, (1):

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial^q u(x,t)}{\partial_+ x^q}+\frac{\partial^q u(x,t)}{\partial_- x^q},\ L \le x \le R,\ 0 \le t \le T. \qquad \ldots[3]$$

together with the initial and zero Dirichlet boundary conditions given by eq. [2], where q is a fractional number, $1 \le q \le 2$.

We recall the left-hande and the right-hande shifted Grünwald estimate (see (2)) to the left and right-handed derivatives, see (3,4).

$$\frac{\partial^q f(x)}{\partial_+ x^q}=\frac{1}{{}_+h^q}\sum_{w=0}^{n_+ +1} g_w f[x-(w-1)h] \qquad \ldots[4]$$

$$\frac{\partial^q f(x)}{\partial_- x^q}=\frac{1}{{}_-h^q}\sum_{w=0}^{n_-} g_w f[x+(w-1)h] \qquad \ldots[5]$$

where n_+, n_- are partial integers, such that:

$$h_+=\frac{n-L}{n_+} \text{ and } h_-=\frac{R-x}{n_-}.$$

where $g_0 = 1$ and

$$g_w = (-1)^w \frac{q(q-1)\ldots(q-w+1)}{w!}, \; w = 1, 2, \ldots \qquad \ldots[6]$$

We divide the x-interval [L,R] into n-subintervals $[x_i, x_{i+1}]$ such that $x_i = L + i\,\Delta x$, $x = 0, 1, \ldots, n$ and $h = \Delta x = \frac{R-h}{n}$.

Also, we divide the t-interval [0,T] into m-subintervals $[t_j, t_{j+1}]$ such that $t_j = j\,\Delta t$, $j = 0, 1, \ldots, m$ and $k = \Delta t = \frac{T}{m}$.

To do this, we substitute $x = x_i$, $t = t_j$ into eq.[1] and [3] and replacing the partial derivatives $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^q u}{\partial x^q}$ with their approximations and using the left-handed and right-handed derivatives in eq.[4], [5].

And $u_{i,j}$ is the numerical solution of FPDE at each (x_i, t_j), $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,m$ such that $u_{i,0} = f(x_i)$ and $u_{0,j} = u_{n,j} = 0$ for $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,m$.
By evaluating the explicit and implicit finite difference methods to solve eq.[1] and eq.[3] at each i and j using the initial-boundary conditions in eq.[2], one can get the numerical solutions of eq.[1] and eq.[3].

2. Stability by Fourier Series Method (Von Neumann′s Method)

This method, developed by Von Neumann during world war II, was first discussed in detail by O′Brien, Hyman and Kaplan in a paper published in 1951, (5).

To expresses an initial line of errors in terms of a finite Fourier series, and consider the growth of a function that reduces this series for t = 0 by a (Variable Separable) method.

The Fourier series can be formulated in terms of sines or consines but the algebra is easier if the complex exponential form is used. That is, with $\sum_n a_n \cos(n\pi x/\ell)$ or $\sum_n b_n \sin(n\pi x/\ell)$ replaced by the equivalent $\sum_n A_n e^{\gamma n\pi x/\ell}$, where $\gamma = \sqrt{-1}$ and ℓ is the interval throughout which the function is defined, and put x =ih, also, t = jk, therefore; changing the notation u(ih,jk) to $u_{i,j}$.
Hence,

$$A_n e^{\gamma n\pi x/\ell} = A_n e^{\gamma n\pi ih/Nh} = A_n e^{\gamma\beta_n ih}, \text{ where } \beta_n = n\pi/Nh \text{ and } Nh = \ell.$$

Denote the errors at the pivotal points along t = 0, between x = 0 and Nh, by $E(ih) = E_i$, i =0, 1, …, N. Then (N+1) the equations:

$$E_i = \sum_{n=0}^{N} A_n e^{\gamma\beta_n ih}, \; i = 0, 1, \ldots, N.$$

are sufficient to determine the (N+1) unknown $A_0, A_1, \ldots, A_n$ uniquely, showing that the initial errors can be expressed in this complex exponential form.

We need only consider the propagation of the error due to a single term, such as $e^{\gamma\beta ih}$. The coefficient A_n is a constant and can be neglected. The investigate the propagation of this error as t increases, we need to find a solution of the finite difference equation which reduces to $e^{j\beta ih}$ when t = jk = 0.

Assume: $E_{i,j} = e^{\gamma\beta x}\, e^{\alpha t} = e^{\gamma\beta ih}\, e^{\alpha jk} = e^{\gamma\beta ih}\, \xi^j$, where $\xi = e^{\alpha k}$, and α, in general, is a complex constant.

This obviously reduces to $e^{\gamma\beta ih}$ when j = 0, the error will not increase as t increases provided $|\xi| \leq 1$, (6).

It should be noted that this method applies only to linear difference equations with constant coefficients, and strictly speaking only to initial value problem with periodic initial data.

The criterion $|\xi| \leq 1$ is necessary and sufficient for two time-level difference equations, (7).

In particle the method often gives useful results even when its application is not fully justified, (6).

3. Stability of the Explicit and Implicit Finite Difference Methods for Solving One-Sided Fractional Partial Differential Equations,(1),(6),(8)

Consider the explicit difference method which results from using the center difference quotient formula for $\left(\dfrac{\partial^2 u}{\partial t^2}\right)$ and using the left-handed shifted Grünwald estimate by eq.[4] for $\left(\dfrac{\partial^q u}{\partial t^q}\right)$, therefore; by substituting that into the FPDE [1], gives us:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{1}{h^q}\sum_{w=0}^{i+1} g_w u_{i-w+1,j}$$

where i =0, 1, …, n – 1 and j = 1, 2, …, m – 1.

The central difference quotient for the second partial derivative is given by:

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \ldots[7]$$

Also from eq.[6], assume $g_1 = -q$, where $1 \le q \le 2$, $i \ne 1$, Hence $g_i \ge 0$ for all value of i. Therefore

$$\sum_{w=0}^{i+1} g_w \le -g_1 = -(-q) = q \qquad \ldots[8]$$

Then the resulting equation can be explicitly solved to give:

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r\sum_{w=0}^{i+1} g_w u_{i-w+1,j} \qquad \ldots[9]$$

where $r = \frac{k^2}{h^q}$.

The difference between analytical and numerical solutions of the difference equation remains bounded as j increases, the error $E_{i,j} = u(h_i,k_j) - u_{i,j}$.

We shall consider the stability conditions under which the finite difference equation [9] is stable, that is; we have to find the stability conditions under which the error $E_{i,j}$ is bounded.

Smith (6) shows that error $E_{i,j}$ can be written in the form:

$E_{i,j} = e^{\gamma\beta ih}\,\xi^j$, where $\xi = e^{\alpha k}$, and α is complex constant, $\gamma = \sqrt{-1}$ …[10]

One can substitute eq.s [8], [10] into [9], to get:

$\xi - 2 + \xi^{-1} - r\,q\,e^{\gamma\beta h(1-w)} \le 0$

Assume: $\theta = \beta h(1 - w)$.

It is easily shown that the equation for ξ is:

$\xi^2 - (2 + rq\,e^{\gamma\theta})\,\xi + 1 = 0$

Let $A = 2 + rq\,e^{\gamma\theta}$, where $|e^{\gamma\theta}| \le 1$.

Hence, the values of ξ are

$$\xi_1 = \frac{A + \sqrt{A^2 - 4}}{2} \text{ and } \xi_2 = \frac{A - \sqrt{A^2 + 4}}{2}.$$

From equation [10], the error will not grow with time if

$|\xi^\gamma| \le 1$, for all real β. …[11]

And eq.[11] is called Von-Neummann′s condition for stability. Thus, we will use the eq.[11] to find the stability condition of the finite difference problem.

For stability; as r, q and β real, and when $A < -1$, then ξ_1 giving stability while ξ_2 giving instability.

When $-1 \le A \le 1$, we get ξ_1 and ξ_2 are complex numbers, hence $\xi_1 = \frac{A + \gamma\sqrt{4 - A^2}}{2}$ and $\xi_2 = \frac{A - \gamma\sqrt{4 - A^2}}{2}$.

Then using Von-Neummann′s condition [11] to prove the eq. [9] is stable.

For – 1 ≤ A ≤ 1, the only useful inequality is A ≤ 1, hence 2 + rq $e^{\gamma\theta}$ ≤ 1, where $|e^{\gamma\theta}| \le 1$.

Therefore; $r \le \frac{-1}{q}$, where 1 ≤ q ≤ 2.

Hence, $|r| \le \frac{1}{2}$.

That leads to the stability condition $|r| \le \frac{1}{2}$.

Now, one can use the similar approach for the implicit finite difference method to solve one-sided FPDEs, the resulting discretization takes the following form:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{1}{h^q}\sum_{w=0}^{i+1} g_w u_{i-w+1,j+1}$$

where i =0, 1, …, n – 1 and j = 1, 2, …, m – 1.

Then to get

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r\sum_{w=0}^{i+1} g_w u_{i-w+1,j+1} \qquad \text{…[12]}$$

Above equation under the same conditions of equation [9] and subsitituting eq.[8] and eq. [10] into eq. [12], one can get:

$\xi - 2 + \xi^{-1} \le rq\, e^{\gamma\theta}\, \xi$, where θ = βh(1 – w).

Hence, the values of ξ are:

$$\xi_1 = \frac{1 + (1 - A)^{\frac{1}{2}}}{A} \text{ and } \xi_2 = \frac{1 - (1 - A)^{\frac{1}{2}}}{A} \text{ where } A = 1 - rqe^{\gamma\theta}.$$

To discusses the stability of equation [12]; by using Von-Neumann′s condition [11]. When A < –1, we get real roots, also, ξ_1 is giving instability while ξ_2 is giving stability for this problem.

Now, when – 1 ≤ A ≤ 1, we get complex numbers, which are $\xi_1 = \frac{1 - \gamma(A - 1)^{\frac{1}{2}}}{A}$ and

$$\xi_2 = \frac{1 + \gamma(A - 1)^{\frac{1}{2}}}{A}.$$

The conditional of stability leads to r ≥ 1 when 1 ≤ q ≤ 2 and $|e^{\gamma\theta}| \le 1$.

Therefore; the finite difference eq. [12] is instable for $r \le \frac{2}{q}$, $1 \le q \le 2$.

4. Stability of the Explicit and Implicit Finite Difference Approximation Methods to Solve Two-Sided Fractional Partial Differential Equation By Stability for Fourier Series Metod, (1), (6), (8)

Take the explicit finite difference approximation method for eq. [3] together with the initial –boundary conditions of eq.[2], then by substituting eq.s [4,5] and [7] into eq. [3], one can write the difference equation as:

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r\left[\sum_{w=0}^{i+1} g_w u_{i-w+1,j} + \sum_{w=0}^{n-i+1} g_w u_{i+w-1,j}\right] \qquad \ldots[13]$$

Next, investigate the stability of above equation, using the same approach as in section 3. One can get:

$\xi - 2 + \xi^{-1} \le 2rq\cos\theta$

Therefore; $\xi^2 - 2(1 + rq\cos\theta)\xi + 1 = 0$, where $\theta = \beta h(1 - w)$.

Assume: $A = 2 + 2rq\cos\theta$, $0 < \cos\theta < 1$.

Hence, the values of ξ are $\xi_1 = \frac{A + \sqrt{A^2 - 4}}{2}$ and $\xi_2 = \frac{A - \sqrt{A^2 - 4}}{2}$.

To discusses the stability of equation [13]; by using eq. [11], thus when $A < -1$, ξ_1 is giving stability while ξ_2 is giving instability.

Also, when $-1 \le A \le 1$, we get ξ_1 and ξ_2 are complex numbers, hence $\xi_1 = \frac{A + \gamma\sqrt{4 - A^2}}{2}$ and $\xi_2 = \frac{A - \gamma\sqrt{4 - A^2}}{2}$.

Also, for $-1 \le A \le 1$, the only useful inequality is $A \le 1$, hence $2 + 2rq\cos\theta \le 1$, where $0 < \cos\theta < 1$.

Therefore; $r \le \frac{-1}{2q}$, where $1 \le q \le 2$.

Hence, $|r| \le \frac{1}{4}$. Thus, the finite difference eq. [13] is stable for $|r| \le \frac{1}{4}$.

Now, carrying similar approach as in the explicit finite difference eq.[13] and by approximation eq. [3] at the points (ih,jk) using implicit difference method becomes:

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r\left[\sum_{w=0}^{i+1} g_w u_{i-w+1,j+1} + \sum_{w=0}^{n-i+1} g_w u_{i+w-1,j+1}\right] \qquad \ldots[14]$$

Now, investigate the stability of eq. [14], by using the same approach in section 3, that leads to:

$(1 - 2rq \cos \theta)\, \xi^2 - 2\xi + 1 \le 0$, where $\theta = (1 - w)\beta h$.

Let $A = 1 - 2rq \cos \theta$, $0 < \cos \theta < 1$.

Hence, the value of ξ are:

$$\xi_1 = \frac{1 + (1-A)^{\frac{1}{2}}}{A} \text{ and } \xi_2 = \frac{1 - (1-A)^{\frac{1}{2}}}{A}.$$

Similarly, using Von-Neumann′s condition, we have:

When $A < -1$, we get real numbers and ξ_1 is giving instability while ξ_2 is giving stability.

Also, when $-1 < A < 1$, then ξ_1 is giving instability while ξ_2 is giving stability. Note that ξ_1 and ξ_2 are real numbers.

Now, when $A > 1$, ξ_1 and ξ_2 are complex numbers, also, they are giving stability and $\xi_1 = \dfrac{1 + \gamma\sqrt{A-1}}{A}$ and $\xi_2 = \dfrac{1 - \gamma\sqrt{A-1}}{A}$.

Now, for stability when $-1 < A < 1$, we have $A > -1$ which is the only useful inequality, hence,

$$r < \frac{1}{q\cos\theta}, \text{ where } 0 < \cos\theta < 1,\ 1 \le q \le 2 \qquad \ldots[15]$$

Therefore, eq. [15] leading to the stability condition $0 < r < \dfrac{1}{2}$, which means that the stability will occur only if $r < \dfrac{1}{2}$.

Conclusions

1. FPDEs are so difficult to be solved analytically; therefore, in most cases, numerical and approximate methods are recommended.
2. The stability results in the FPDE case are a generalization and unification for the corresponding results in the classical hyperbolic PDEs.
3. The explicit finite difference method using the shift Grünwald method to solve the one-sided FPDEs is conditionally stable while the implicit of this scheme is instable.

4. The explicit and implicit finite difference method using the shift Grünwald method to solve the two-sided FPDEs is conditionally stable.
5. The stability results for implicit FPDEs are more realizable than explicit FPDEs.

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